Inequalities

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Basic Principle of Inequalities: For any real number x, we have

$$x^2 \geq 0\ , \text{ with equality if and only if } x = 0\ .$$

Example. For any two positive real numbers x and y, we have $(x-y)^2 \geq 0$, and so $x^2 + y^2 - 2xy \geq 0$. Writing this as

$$x^2 + y^2 + 2xy \geq 4xy,$$

we get

$$\left(\frac{x+y}{2}\right)^2 \geq xy.$$

Taking the square root of both sides yields

$$\frac{x+y}{2} \geq \sqrt{xy}\ .$$

where by convention, $\sqrt{\cdot}$ denotes the *positive* square root. This inequality has a special name.

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality:

For any two positive real numbers x and y, we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

The quantity of the LHS is called the *arithmetic mean* of the two numbers x and y. The quantity of the LHS is called the *geometric mean* of the two numbers x and y. They can be regarded as providing two different ways of "averaging" a pair of numbers.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of two positive quantities whose *product* is fixed occurs when both are equal.
- The *maximum* value of the *product* of two positive quantities whose *sum* is fixed occurs when both are equal.
- A geometric interpretation of this result is that in any right-angled triangle, the median corresponding to the hypotenuse is bigger than the altitude corresponding to hypothenuse.

Example. Find the minimum of $x + \frac{5}{x}$, where x is positive.

Solution. By the AM-GM inequality,

$$\begin{aligned} x + \frac{5}{x} &\geq 2\sqrt{(x) \cdot \left(\frac{5}{x}\right)} \\ &= 2\sqrt{5}\,. \end{aligned}$$

The minimum occurs when $x = \frac{5}{x}$, i.e., when $x = \sqrt{5}$.

Example. Prove that for any positive numbers a, b and c we have

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. By the AM-GM inequality we have

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \frac{b+c}{2} \geq \sqrt{bc}, \quad \frac{c+a}{2} \geq \sqrt{ca}$$

If we multiply these three inequalities we find

$$\frac{(a+b)(b+c)(c+a)}{8} \geq \sqrt{(ab)(bc)(ca)} = abc$$

and this finishes our proof.

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality (more than two variables):

Suppose we have n positive real numbers $x_1, x_2, \ldots, x_n$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of positive quantities whose *product* is fixed occurs when all are equal.
- The *maximum* value of the *product* of positive quantities whose *sum* is fixed occurs when all are equal.

Example.

Minimize $x^2 + y^2 + z^2$ subject to $x, y, z > 0$ and $xyz = 1$.

Solution. By AM-GM,

$$\begin{aligned} x^2 + y^2 + z^2 &\geq 3\sqrt[3]{x^2 \cdot y^2 \cdot z^2} \\ &= \sqrt[3]{(xyz)^2} \\ &= 1\ . \end{aligned}$$

The minimum occurs when $x^2 = y^2 = z^2$, i.e., when $x = y = z = 1$.

Example.

Minimize $\frac{6x}{y}+\frac{12y}{z}+\frac{3z}{x}$ for $x,y,z>0$.

Solution. By AM-GM,

$$\frac{6x}{y}+\frac{12y}{z}+\frac{3z}{x} \geq 3\sqrt[3]{\frac{6x}{y}\cdot\frac{12y}{z}\cdot\frac{3z}{x}} = 3\sqrt[3]{6\cdot 12\cdot 3} = 3\cdot 6 = 18\ .$$

The minimum occurs if and only if $\frac{6x}{y}=\frac{12y}{z}=\frac{3z}{x}$, i.e., if and only if $x=t$, $y=t$ and $z=2t$ for some positive number t.

Example.

Maximize $xy\,(72-3x-4y)$, where $x,y>0$ and $3x+4y<72$.

Solution. We seek to maximize the product of three positive quantities. Note that the sum of the three quantities is equal to

$$x+y+(72-3x-4y) = 72-2x-3y\ .$$

This is NOT a constant! However, we can rearrange the product as

$$\frac{1}{12}\,(3x)\,(4y)\,(72-3x-4y)$$

Thus by AM-GM, the maximum occurs when $3x=4y=72-3x-4y$, i.e., when $3x=72-6x$. This yields $9x=72$, or $x=8$. Thus $y=6$ and the maximum value is $\frac{1}{12}\cdot(24)^3=1152$.

Example.

Let a be a positive constant. Minimize $x^2 + \frac{a}{x}$, where $x > 0$.

Solution. We seek to minimize the sum of two quantities. Note that the product of the two quantities is equal to ax – this is NOT a constant. However, we can rearrange the sum as

$$x^2 + \frac{a}{2x} + \frac{a}{2x} \ .$$

Thus using AM-GM,

$$x^2 + \frac{a}{2x} + \frac{a}{2x} \geq 3\sqrt[3]{x^2 \cdot \frac{a}{2x} \cdot \frac{a}{2x}} = 3\sqrt[3]{\frac{a^2}{4}} = 3\left(\frac{a}{2}\right)^{\frac{3}{2}} \ .$$

The minimum occurs when $x^2 = \frac{a}{2x} = \frac{a}{2x}$, i.e. when $x = \sqrt[3]{\frac{a}{2}}$.

Two More "Averages":

The **Harmonic Mean** of n numbers $x_1, x_2, \ldots, x_n$ is given by

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

and their **Root-Mean-Square** is given by

$$\mathrm{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}\ .$$

If all the numbers $x_1, x_2, \ldots, x_n$ are positive, then we have

$$\min\{x_1, \ldots, x_n\} \leq \mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM} \leq \mathrm{RMS} \leq \max\{x_1, \ldots, x_n\}$$

with equality in each case if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Special case: for two positive numbers x and y

$$\min\{x, y\} \leq \frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}} \leq \max\{x, y\}\ .$$

Exercise: Prove the above special case (*all* inequalities)!

Looking at the AM-HM inequality, we have AM $\geq$ HM, or

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

This can be rearranged into the form

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2 \,,$$

with equality if and only if the numbers $x_1, x_2, \ldots, x_n$ are all equal.

Example: "Nesbitt's Inequality".

Prove that for positive numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2} \,.$$

Solution. Write the LHS as

$$\begin{aligned}
&\frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3 \\
&\qquad = (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3 \\
&\qquad = \frac{1}{2}[(a+b) + (b+c) + (a+c)]\left[\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right] - 3 \\
&\qquad \geq \frac{1}{2}(9) - 3 = \frac{3}{2}
\end{aligned}$$

where we have used the HM-AM inequality with $n = 3$:

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3^2$$

with $x = a+b$, $y = b+c$, $z = a+c$.

Sometimes we can be asked to prove an inequality regarding the *sides lengths of a triangle*. Here, the side lengths a, b, c (aside from being positive) must satisfy the so-called *triangle inequalities*:

$$a + b > c \,; \quad b + c > a \,; \quad c + a > b \,;$$

Example.

Let a, b, c be the side lengths of a triangle. Prove that

$$a^2 + b^2 + c^2 < 2(ab + bc + ca).$$

Solution.

Note that for example, if $a = 5$ and $b = c = 1$, we have

$$a^2 + b^2 + c^2 = 27 \,; \quad 2(ab + bc + ca) = 22.$$

and the result does *not* hold. Therefore, it is important that we use the information that a, b, c satisfy the triangle inequalities.

Writing the triangle inequality $a + b > c$ as $c - b < a$ and squaring, we obtain $(c - b)^2 < a^2$. Doing this for each triangle inequality yields

$$(c - b)^2 < a^2$$

$$(a - b)^2 < c^2$$

$$(c - a)^2 < b^2$$

Adding these three inequalities, and simplifying, yields the result (**Exercise:** check this!).

Exercises.

(1) Let $x, y > 0$. Find the minimum of
$$\frac{50}{x} + \frac{20}{y} + xy.$$

(2) If $x > y > 0$, find the minimum of
$$x + \frac{8}{y(x-y)}.$$

(3) Prove that for any positive real numbers a, b, c we have
$$(a+9b)(b+9c)(c+9a) \geq 216abc.$$

(4) Prove that for positive real numbers x, y, z,
$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$
and determine when equality occurs.

(5) Find the positive number whose square exceeds its cube by the greatest amount.

(6) Prove that for positive real numbers x, y, z,
$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

(7) The sum of a number of positive integers is 2016. Determine the maximum value their product could have.

For further reading, click here: Wikipedia entry on AM-GM